Top Equations in the Field

What are the key equations that changed the control field? Reference [1] makes interesting reading about the equations that changed the world, including such gems as the Gaussian/normal distribution, the Fourier transform (with which I associate the Laplace transform), and Shannon's quantification of information, all of which play important roles in the control field. But what else? Here is an attempt to list the top equations in the field. Do you agree? Join the conversation by e-mailing me at jhow@mit.edu.

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» **Lyapunov's Stability Theorem**

Let $x = 0$ be an equilibrium point for a dynamic system $\dot{x} = f(x)$ and $D \subset R^n$ be a domain containing $x = 0$. Let $V: D \to R$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\},$$
$$\dot{V}(x) \leq 0 \text{ in } D.$$

Then $x = 0$ is Lyapunov stable. Moreover, if $\dot{V}(x) < 0$ in D, then $x = 0$ is asymptotically stable [2, p. 100].

These expressions provide a means for analyzing the stability of linear and nonlinear systems without having to solve for the trajectories explicitly. A continuously differentiable function satisfying these equations is known as a Lyapunov function and is essentially a generalized energy function for a system. Numerous extensions of Lyapunov's theorem exist, including Lasalle's theorem, which relaxes the negative definiteness requirements on $\dot{V}(x)$.

» **Sensitivity Constraints**

Bode's sensitivity integral

$$\int_0^\infty \ln|S(j\omega)|d\omega = \pi \sum_{i=1}^{N_p} R(p_i) > 0$$

gives a constraint on the sensitivity function $S(s)$ for a stable closed-loop system with a loop transfer function $L(s)$ for which there are at least two more poles than zeros and N_p right-half plane poles at locations p_i [3, p. 166]. The expression captures the tradeoff that exists in the possible reduction in the sensitivity in different frequency regions, thereby conveying a limitation on performance that can be achieved.

The constraint leads to the "push-pop" phenomenon (more commonly know as the water bed effect) in which large reductions (push) in $|S|$ in one frequency region to improve performance lead to increases in $|S|$ (pop) in other regions, possibly leading to issues of poor robustness. Numerous extensions of this foundational equation have been developed for unstable and nonminimum phase systems. The concept has broad applicability, and similar results have been shown to hold for biological systems [4].

» **Lagrange Multipliers**

In optimal control problems, the equation for the Lagrangian function is typically written as

$$L(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = F(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^T \boldsymbol{f}(\mathbf{x}, \mathbf{u}),$$

which adjoins the cost function F with constraint equations $\mathbf{f}(\mathbf{x}, \mathbf{u})$ using the Lagrange multipliers $\boldsymbol{\lambda}$.

Lagrange multipliers provide a principled means of solving constrained optimization problems as unconstrained (and thus simpler) problems. The Lagrange multipliers associated with a constrained optimization can be interpreted as the price of a constraint requirement, which can provide important insights in system-level design and economics [5]. The generalization of Lagrange multipliers to a costate yields a closed-form solution of optimal control problems [6].

» **Optimal Control**

The Hamilton–Jacobi–Bellman (HJB) equation

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$$0 = J_t^*(x(t),t) + \min_{u(t)\in\mathcal{U}} \{g(x(t),u(t),t) + J_x^*(x(t),t)\,a(x(t),u(t),t)\}.$$

This is a nonlinear partial differential equation in $J^*(x(t),t)$ that is solved backward in time from t_f for a problem for which the integrand of the cost function is given by $g(x(t),u(t),t)$ and the system dynamics are $\dot{x} = a(x,u,t)$. The equation builds on the principle of optimality [7] and Pontryagin's maximum principle [8] and is one of the foundational results in the field of optimal control. Although the differential equation is very difficult to solve, the approach provides a principled means of determining the best possible solution to a given optimal control problem and thus is widely used and approximated [9].

An important example where the HJB equation can be solved analytically is the case where system dynamics are linear with Gaussian process and measurement noises and the cost is quadratic. In this case, the solution of the HJB equation leads to the celebrated linear-quadratic-Gaussian controller.

» Riccati Equation

The matrix version of the Riccati equation is written as

$$-\dot{P}(t) = P(t)A(t) + A(t)^T P(t) + Q(t) - P(t)B(t)R(t)^{-1} \times B(t)^T P(t),$$

which is named after Count Jacopo Franceso Riccati (1676–1754). The general form is a first-order ordinary differential equation that is quadratic in the unknown parameter, in this case $P(t)$. The steady-state version of this equation for continuous $(\dot{P} = 0)$ and discrete $(P_{k+1} = P_k)$ systems are known as the continuous algebraic Riccati equation (CARE) and discrete algebraic Riccati equation (DARE), respectively.

CARE and DARE are nonlinear equations that mainly arise when dealing with infinite-horizon filtering or optimal control problems. A solution to CARE/DARE can be obtained by matrix factorizations or by iterating on the Riccati equation. This foundational equation appears regularly because it provides the solution to the linear-quadratic-regulator problem, the Kalman filter problem, the $\mathcal{H}_\infty$-control-design problem, as well as many others [3].

» Bayes' Rule

Bayes' rule is written as

$$f_{x\,|\,y}(x|y) = \frac{f_{y\,|\,x}(y|x) f_x(x)}{f_y(y)},$$

which, given the joint probability density function $f_{x,y}(x,y)$ for random variables x and y, provides a means of relating the conditional probabilities of x and y. The rule was named after Thomas Bayes, but his work was significantly updated by Richard Price before being posthumously presented to the Royal Society in 1763 [10]. Bayes' rule provides a direct way to relate the prior and posterior probability density functions of a quantity after measurements or observations have been made.

As a result, Bayes' rule is a foundational result for estimation and reinforcement learning.

As a result, Bayes' rule is a foundational result for estimation and reinforcement learning. The importance of the result is perhaps best captured by the quote from Sir Harold Jeffreys, who wrote that "This theorem (due to Bayes) is to the theory of probability what Pythagoras's theorem is to geometry" [11, p. 31].

» Least Squares

The solution to the least-squares problem $\min_x \| y - Ax \|_2$ can be written as

$$x = A^\dagger y,$$

where $A^\dagger$ is the pseudoinverse or Moore–Penrose generalized inverse of A. While the result is often attributed to Gauss in 1809, the original developer is open to some dispute [12].

Least-squares methods are ubiquitous and play an important role in fitting data, such as in system-identification methods. Also note that if $A^T A$ is positive definite and thus has a unique inverse, $A^\dagger = (A^T A)^{-1} A^T$, which is immediately recognizable as being of the same form as the steady-state gain for a Kalman filter [13].

» Small Gain

Consider a system with a stable loop-transfer function $L(s)$. Then the closed-loop system is stable if

$$\| L(j\omega) \| < 1 \quad \text{for all } \omega,$$

where $\| \cdot \|$ denotes any matrix norm satisfying $\| AB \| \le \| A \| \cdot \| B \|$ [3, p. 150]. While often conservative because it ignores phase information, the test provides a launching point for robust control. The form given here generalizes to enable the input–output stability analysis of nonlinear time-varying systems [14]. The approach also leads to extensions of absolute stability theory (with multipliers) [15], ultimately leading to K_m analysis [16] (closely related to the complex structured singular value μ) [17].

» ***S*-Procedure**

Let $F_i, i \in \{0,\ldots,p\}$, be quadratic functions of the variable $x \in \mathcal{R}^n$ of the form

$$F_i(x) = x^T T_i x + 2u_i^T x + v_i,$$

where $T_i = T_i^T$. The goal is to consider the following condition on the F_is

$$F_0(x) \geq 0 \text{ for all } x \text{ such that } F_i(x) \geq 0,\ i = 1,\ldots,p. \quad (1)$$

If there exist $\tau_i \geq 0$ such that, for all x,

$$F_0(x) - \sum_{i=0}^{p} \tau_i F_i(x) \geq 0, \quad (2)$$

then (1) holds. This considers problems of the form in which some quadratic function is positive whenever some other quadratic functions are all positive [18]. The importance of the S-procedure is that it provides a means of verifying (1) using (2), which is useful in that (2) is generally in a much simpler form than (1).

The procedure dates to Lur'e and Postnikov [19], with more recent significant contributions by Yakubovich [20]. The procedure is particularly important for composing linear matrix inequalities, which play a very important role in the robust control literature [18].

THE NEXT EQUATION

What do you think is missing? E-mail me at jhow@mit.edu with what should be added to the list.

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Sensitivity Reduction

The problem of sensitivity reduction by feedback is formulated as an optimization problem and separated from the problem of stabilization. Stable feedback schemes obtainable from a given plant are parameterized. Salient properties of sensitivity reducing schemes are derived, and it is shown that plant uncertainty reduces the ability of feedback to reduce sensitivity. The theory is developed for input-output systems in a general setting of Banach algebras, and then specialized to a class of multivariable, time-invariant systems characterized by n x n matrices of H∞ frequency response functions, either with or without zeros in the right half-plane. … we shall be concerned with the effects of feedback on uncertainty, where uncertainty occurs either in the form of an additive disturbance d at the output of a linear plant P, or an additive perturbation in P representing "plant uncertainty." We shall approach this subject from the point of view of classical sensitivity theory, with the difference that feedbacks will not only reduce but actually optimize sensitivity in an appropriate sense.

— George Zames

"Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses,"

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